METRIC AND ULTRAMETRIC SPACES OF RESISTANCES

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Abstract. Given an electrical circuit each edge e of which is an isotropic conductor with a monomial conductivity function

$$y_e^* = y_e^r / \mu_e^s.$$

In this formula, y_e is the potential difference and y_e^* current in e, while μ_e is the resistance of e; furthermore, r and s are two strictly positive real parameters common for all edges. In particular, the case $r = s = 1$ corresponds to the standard Ohm law, while $r = 0.5$ is the standard square law of resistance typical for hydraulics or gas dynamics.

For every two nodes a, b of the circuit, the effective resistance $\mu_{a,b}$ is well-defined and for every three nodes a, b, c the following inequality holds:

$$\mu_{a,b}^{s/r} \leq \mu_{a,c}^{s/r} + \mu_{c,b}^{s/r}.$$

It obviously implies the standard triangle inequality $\mu_{a,b} \leq \mu_{a,c} + \mu_{c,b}$ whenever $s \geq r$.

The equality takes place if and only if each path between a and b contains c.

One gets several interesting metric and ultrametric spaces playing with parameters r and s; in particular,

- (i) the effective Ohm resistance for $r(t) = s(t) \equiv 1$;
- (ii) the length of a shortest path for $r(t) = s(t) \to \infty$;
- (iii) the inverse width of a bottleneck path for $r(t) \equiv 1, s(t) \to \infty$;
- (iv) the inverse capacity (maximum flow per unit time) for $r(t) \to 0, s(t) \equiv 1$;

between any pair of terminals a and b, as $t \to \infty$. In all four cases the limits $\mu_{a,b} = \lim_{t\to\infty} \mu_{a,b}(t)$ exist for all pairs a, b and the metric inequality

$$\mu_{a,b} \leq \mu_{a,c} + \mu_{c,b}$$

holds for all triplets a, b, c, since $s(t) \geq r(t)$ for any sufficiently large t.
Moreover, the stronger ultrametric inequality

$$\mu_{a,b} \leq \max(\mu_{a,c}, \mu_{c,b})$$

holds for all triplets a, b, c in examples (iii) and (iv), since in these two cases $s(t)/r(t) \to \infty$, as $t \to \infty$.